Theory of equations

1. **(a)** Let $p(x) = x^4 + 4x^3 - 8x^2 - 1$, then $p'(x) = 4x^3 + 12x^2 - 16x = 4x(x + 4)(x - 1)$

$\therefore$ $x = -4, 0, 1$ are roots of the equation $p'(x) = 0$

x	$-\infty$	-4	0	1	$+\infty$
p(x)	$+\infty$	-9	-1	-4	$+\infty$

$\therefore$ There is one real root with $x < -4$ and another with $x > 1$.

There is no real root with $-4 < x < 1$.

Since $p(-5) < 0$, $p(-6) > 0$ and $p(1) < 0$ and $p(2) > 0$.

There are two real roots α ,β for $p(x) = 0$ with $-6 < \alpha < -5$ and $1 < \beta < 2$.

(b) Let $p(x) = 8x^5 - 5x^4 - 40x^3 - 50$, then $p'(x) = 40x^4 - 20x^3 - 120x^2 = 20x^2(2x + 3)(x - 2)$

$\therefore$ $x = -3/2, 0, 2$ are roots of the equation $p'(x) = 0$

x	$-\infty$	$-3/2$	0	2	$+\infty$
p(x)	$-\infty$	$-$	$-$	$-$	$+\infty$

$\therefore$ There is one real root with $x > 2$.

Since $p(3) < 0$, $p(4) > 0$,there is one real root α for $p(x) = 0$ with $3< \alpha < 4$.

2. **(a)** Let $p(x) = x^4 + 2x^2 + 3x - 1$.

The number of sign change for $p(x)$ is 1 .

By the Decartes' rule of sign, there is at most one positive root.

$p(-x) = x^4 + 2x^2 - 3x - 1$.

The number of sign change for $p(-x)$ is 1 .

By the Decartes' rule of sign, there is at most one negative root.

There are at most two real roots.

Since $\deg[p(x)] = 4$, there is at least 2 complex roots.

$p(-2) < 0$, $p(-1) < 0$ and $p(0) < 0$, $p(1) > 0$.

$\therefore$ There are 2 complex roots and 2 real roots α ,β for $p(x) = 0$ with

$-2 < \alpha < -1$ and $0< \beta < 1$.

(b) Let $p(x) = x^5 - 2x^3 + x - 10$, then $p'(x) = 5x^4 - 6x^2 + 1 = \left(5x^2 - 1\right)\left(x^2 - 1\right)$

$\therefore$ $x = -1, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 1$ are roots of the equation $p'(x) = 0$

x	$-\infty$	-1	$-\frac{1}{\sqrt{5}}$	$\frac{1}{\sqrt{5}}$	1	$+\infty$
p(x)	$-\infty$	$-$	$-$	$-$	$-$	$+\infty$

$\therefore$ There is one real root with $x > 1$.

Since $p(1) < 0$, $p(2) > 0$,there is one real root α for $p(x) = 0$ with $1< \alpha < 2$.

3. α, β, γ are the roots of the equation $x^3 - px + q = 0$ …. (1)

$\alpha + \beta + \gamma = 0$ …. (2) $\qquad \alpha\beta + \beta\gamma + \gamma\alpha = -p$ …. (3) $\qquad \alpha\beta\gamma = -q$ …. (4)

Now, $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta = \alpha^2 + \beta^2 + \gamma^2 - \gamma^2 - \dfrac{2\alpha\beta\gamma}{\gamma} = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) - \gamma^2 - \dfrac{2\alpha\beta\gamma}{\gamma}$

$$= 0^2 - 2(-p) - \gamma^2 - \frac{2(-q)}{\gamma} = 2p - \gamma^2 + \frac{2q}{\gamma} \quad \text{, by (1) - (4) .}$$

$\therefore \quad (\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$ are roots of a new equation formed by putting:

$y = 2p - x^2 + \dfrac{2q}{x} \Leftrightarrow xy = 2px - x^3 + 2q \Leftrightarrow (x^3 - px + q) + xy - px = 3q \Leftrightarrow xy - px = 3q \Leftrightarrow x = \dfrac{3q}{y-p}$ …. (5)

Substitute (5) in (1), $\quad \left(\dfrac{3q}{y-p}\right)^3 - p\left(\dfrac{3q}{y-p}\right) + q = 0 \quad \Leftrightarrow y^3 - 6py^2 + 9p^2y + (27q^2 - 4p^3) = 0$ ….(6)

The product of roots of (6) $D = (\alpha - \beta)^2 (\beta - \gamma)^2 (\gamma - \alpha)^2 = -(27q^2 - 4p^3) = 4p^3 - 27q^2$ …. (7)

Since (1) is of degree 3 , the condition of roots are shown below:

(i) 3 real and distinct roots $\Leftrightarrow$ $D > 0$.

(ii) 1 real double root and one other real root or 1 triple root $\Leftrightarrow$ $D = 0$.

(iii) 2 complex roots and 1 real roots $\Leftrightarrow$ $D < 0$.

(w.l.o.g. let $\alpha = a + bi$, $\beta = a - bi$, $\gamma = c$ and substitute in (7))

$\therefore$ The roots of (1) should be real $\Leftrightarrow$ $D \geq 0$ $\Leftrightarrow$ $4p^3 - 27q^2 \geq 0$.

4. $x^2 - 3x + 4 = \lambda(1 + 2x) \quad \Leftrightarrow \quad x^2 - (3 + 2\lambda)x + (4 - \lambda) = 0$

This quadratic equation has real roots $\Leftrightarrow$ $\Delta = (3 + 2\lambda)^2 - 4(4 - \lambda) \geq 0$

$\Leftrightarrow 4\lambda^2 + 16\lambda \geq 0$

$\Leftrightarrow \quad \lambda \leq \dfrac{4 - \sqrt{21}}{2} \quad$ or $\quad \lambda \geq \dfrac{4 + \sqrt{21}}{2}$

$x^2 - 3x + 4 = \lambda(1 + 2x)$

$\Leftrightarrow \begin{cases} y = x^2 - 3x + 4 & \text{....(1)} \\ y = \lambda(1 + 2x) & \text{....(2)} \end{cases}$

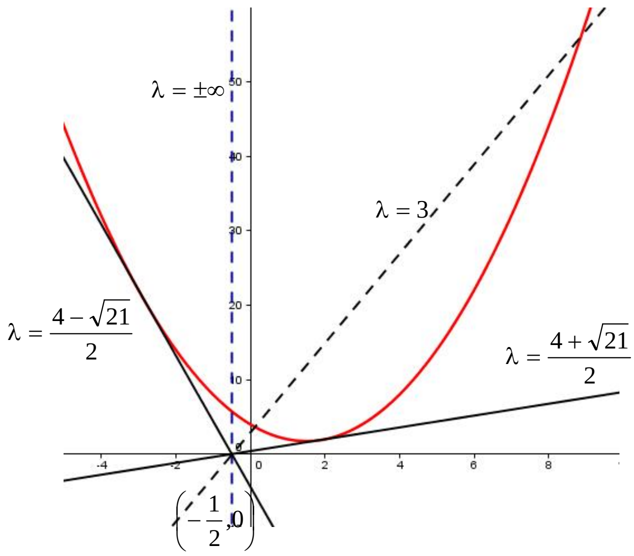
(2) is a pencil of straight lines passing through

the point $\left(-\dfrac{1}{2}, 0\right)$.

(1) and (2) cut each other only when

$\lambda \leq \dfrac{4 - \sqrt{21}}{2} \quad$ or $\quad \lambda \geq \dfrac{4 + \sqrt{21}}{2}$

as shown in the graph.



When $\lambda = 3$, (2) becomes $y = 3 + 6x$.

When $\dfrac{9 - \sqrt{77}}{2} < x < \dfrac{9 + \sqrt{77}}{2}$ (0.11 < x < 8.89), $3 + 6x$ greater than $(x^2 - 3x + 4)$.

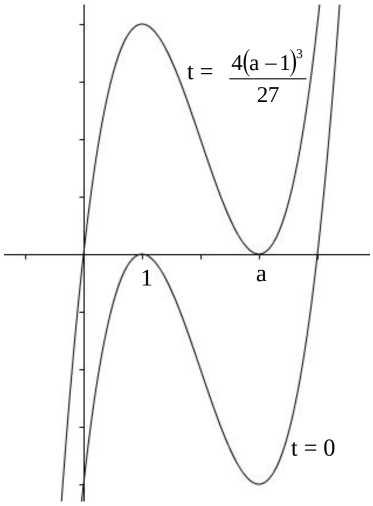
5. Let $y = (x - 1)^2 (x - a) + t \quad (a > 1)$

$y' = (x - 1)(3x - 2a - 1) \quad , \quad y'' = 2(3x - a - 2)$

For critical values, $y' = 0$. $\therefore$ $x = 1$ or $x = \frac{2a+1}{3}$.

$y''\big|_{x=1} = 2(1-a) < 0$ since $a > 1$. $\therefore$ y is a max. when $x = 1$, $y_{max} = t$.

$y''\big|_{x=\frac{2a+1}{3}} = 2(a-1) > 0$ since $a > 1$. $\therefore$ y is a min. when $x = \frac{2a+1}{3}$, $y_{min} = -\frac{4(a-1)^3}{27} + t$



As $x \to -\infty$, $y \to -\infty$ and as $x \to +\infty$, $y \to +\infty$.

The equation $y = 0$ has three real roots $\Leftrightarrow$ $y_{max} > 0 \wedge y_{min} < 0$

$\Leftrightarrow$ $t > 0 \wedge -\frac{4(a-1)^3}{27} + t < 0$ $\Leftrightarrow$ $0 < t < \frac{4(a-1)^3}{27}$

$y = (x-1)^2(x-a) + t$ $\Leftrightarrow$ $y = x^3 - (2+a)x^2 + (1-2a)x + (a+t)$

$\therefore$ $\alpha + \beta + \gamma = 2 + a$

$\alpha\beta + \beta\gamma + \gamma\alpha = 1 - 2a$

$\alpha\beta\gamma = a + t$

$\therefore$ $\beta\gamma + \gamma\alpha + \alpha\beta - 2(\alpha + \beta + \gamma) + 3$

$= 1 - 2a - 2(2 + a) + 3 = 0$

6. Let $x = \frac{p}{q}$ be a rational root of $ax^2 + bx + c = 0$, where $p, q \in \mathbb{Z}$, H.C.F $(p,q) = 1$, $p, q \neq 0$.

By the rational zero theorem, $p|c$ and $q|a$.

Since a, c are odd and $p, q \neq 0$, p, q are odd.

However $ax^2 + bx + c = 0$ $\Rightarrow$ $a\left(\frac{p}{q}\right)^2 + b\left(\frac{p}{q}\right) + c = 0$ $\Rightarrow$ $ap^2 + bpq + cq^2 = 0$ $(q \neq 0)$

There is a contradiction since ap^2, bpq, cq^2 are all odd so their sum is also odd, and must not be equal to 0.

7. **(a)** $P(x) \equiv a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$

$P(p) - P(q) = a_0(p^n - q^n) + a_1(p^{n-1} - q^{n-1}) + \ldots + a_{n-1}(p-q)$

$= (p-q)\left[a_0(p^{n-1} + p^{n-2}q + \ldots + q^{n-1}) + a_1(p^{n-2} + p^{n-3}q + \ldots + q^{n-2}) + \ldots + a_{n-1}\right]$

Since $a_0, a_1, \ldots, a_n, p, q$ are integers, the second factor in the above expression is an integer.

The first factor $(p - q)$ is an even integer if p, q are both even integers (or both odd integers).

$\therefore$ $P(p) - P(q)$ is even.

(b) Suppose, on contrary, $x = q$ is an integral root of $P(x) = 0$. Then $P(q) = 0$.

(i) If q is odd, then $P(1) - P(q)$ is even by **(a)**.

But $P(1) - P(q) = P(1) - 0 = P(1)$, which is odd by given.

(ii) If q is even, then $P(0) - P(q)$ is even by **(a)**.

But $P(0) - P(q) = P(0) - 0 = P(1)$, which is odd by given.

In both cases, there is a contradiction.

8. If α is the root of the equation $ax^2 + bx + c = 0$, then $a\alpha^2 + b\alpha + c = 0$ and $a\alpha^2 = -b\alpha - c$ ….(1)

$A\alpha^2 + B\alpha + C$ is rational $\Leftrightarrow$ $a(A\alpha^2 + B\alpha + C) = A(-b\alpha - c) + Ba\alpha + Ca$, by (1)

$= (-Ab + Ba)\,\alpha - Ac + Ca$ is rational

$\Leftrightarrow$ $-Ab + Ba = 0$, since α is irrational and all other constants are rational .

$\Leftrightarrow$ $Ab = Ba$ …. (2)

From (1), $a\alpha^3 = -b\alpha^2 - c\alpha$ …. (3)

$A\alpha^3 + B\alpha^2 + C\alpha$ is rational $\Leftrightarrow$ $Aa\alpha^3 + Ba\alpha^2 + Ca\alpha = A(-b\alpha^2 - c\alpha) + Ba\alpha^2 + Ca\alpha$, by (2)

$= (Ba - Ab)\,\alpha^2 + (Ca - Ac)\,\alpha = (Ca - Ac)\,\alpha$ is rational , by (3)

$\Leftrightarrow$ $Ca - Ac = 0$, since α is irrational and all other constants are rational .

$\Leftrightarrow$ $Ac = Ca$

Let $p(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$. Then by Division Algorithm,

$p(x) = (ax^2 + bx + c)\,q(x) + mx + n$, where m, n are rational constants.

Now, $p(x)$ is rational $\Leftrightarrow$ $(a\alpha^2 + b\alpha + c)\,q(\alpha) + m\alpha + n$ is rational .

$\Leftrightarrow$ $m\alpha + n$ is rational , given that $a\alpha^2 + b\alpha + c = 0$

$\Leftrightarrow$ $m = 0$, since α is irrational

$\Leftrightarrow$ $p(x) = (ax^2 + bx + c)\,q(x) + n$, that is, the remainder is independent of x .

9. Let $y = p(x) = x^3 - px + q$ then $p'(x) = 3x^2 - p$, $p''(x) = 6x$

Since $\lim_{x\to+\infty} p(x) = +\infty$, $\lim_{x\to-\infty} p(x) = -\infty$, $p(x) = 0$ has at least one real root .

For stationary points, $p'(x) = 0$, $3x^2 - p = 0$ $\Rightarrow$ $x = \pm\sqrt{\frac{p}{3}}$ …. (1)

From (1), There are two turning points if $p > 0$.

Since $p''\left(+\sqrt{\frac{p}{3}}\right) > 0$, y is a min when $x = +\sqrt{\frac{p}{3}}$. $y_{min} = q - \frac{2p}{3}\sqrt{\frac{p}{3}}$,

Since $p''\left(-\sqrt{\frac{p}{3}}\right) > 0$, y is a max when $x = -\sqrt{\frac{p}{3}}$ $y_{max} = q + \frac{2p}{3}\sqrt{\frac{p}{3}}$,

$\therefore$ $p(x) = 0$ has 3 real roots (or 1 double real root and 1 real root) $\Leftrightarrow$ $y_{min} \le 0$ and $y_{max} \ge 0$

$\Leftrightarrow$ $q - \frac{2p}{3}\sqrt{\frac{p}{3}} \le 0$ and $q + \frac{2p}{3}\sqrt{\frac{p}{3}} \ge 0$ $\Leftrightarrow$ $\left(q - \frac{2p}{3}\sqrt{\frac{p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{p}{3}}\right) \le 0$

$\Leftrightarrow$ $4p^3 \ge 27q^2$

y has only one real root $\Leftrightarrow$ $[y_{min} > 0$ and $y_{max} > 0]$ or $[y_{min} < 0$ and $y_{max} < 0]$

$\Leftrightarrow$ $\left(q - \frac{2p}{3}\sqrt{\frac{p}{3}}\right)\left(q + \frac{2p}{3}\sqrt{\frac{p}{3}}\right) > 0$ $\Leftrightarrow$ $4p^3 < 27q^2$.

(a) $x^3 - 2x + 7 = 0$, Since $4p^3 = 4(2)^3 = 32$, $27q^2 = 27(7)^2 = 1323$, $\therefore 4p^3 < 27q^2$.

The equation has only one real root.

(b) $3x^3 + 4x - 2 = 0$, Since $4p^3 = 4(-4/3)^3 = -256/27$, $27q^2 = 27(-2/3)^2 = 12$, $\therefore 4p^3 < 27q^2$.

The equation has only one real root.

(c) $4x^3 - 7x + 3 = 0$. Since $4p^3 = 4(7/4)^3 = 343/16$, $27q^2 = 27(3/4)^2 = 243/16$, $\therefore 4p^3 > 27q^2$.

The equation has 3 real roots .

10. **(a)** $x^3 + Px^2 + Qx + R = 0$ …. (1)

Put $x = X + k$, where k is a constant .

$(X + k)^3 + P(X + k)^2 + Q(X + k) + R = 0$ …. (2)

In (2) , the coeff. of x^2-term is $(3k + P)$.

By putting $3k + P = 0$ or $k = -\frac{P}{3}$, then the transformation $x = X - \frac{P}{3}$ can transform (1) into (2) where x^2-term disappears.

(b) $x^3 - 15x = 126$ …. (3) $x = y + z$ …. (4)

(4) $\downarrow$ (3), $(y + z)^3 - 15(y + z) = 126 \Leftrightarrow y^3 + z^3 + 3y^2z + 3yz^3 - 15(y + z) = 126$

$\Leftrightarrow y^3 + z^3 + (3yz - 15)(y + z) = 126 \Leftrightarrow y^3 + z^3 + (3yz - 15)x = 126$ …. (5)

If further we choose $3yz - 15 = 0$, or $yz = 5$ …. (6)

(5) becomes : $y^3 + z^3 = 126$ …. (7)

From (6), $(y^3)(z^3) = 125$ …. (8)

From (7) and (8), y^3 and z^3 are roots of a quadratic equation $t^2 - 126t + 125 = 0$ …. (9)

If y^3 and z^3 are both real and y_0, z_0 are their real roots, possible values for y, z are y_0, ωy_0, $\omega^2 y_0$ and $z_0, \omega z_0, \omega^2 z_0$.

By (6) , the product yz must be real. Therefore the only combinations consistent with this give, as the three roots of the cubic, by (4) : $y_0 + z_0$, $\omega y_0 + \omega^2 z_0$, $\omega^2 y_0 + \omega z_0$.

Similarly for the eq. $x^3 - 15x = 126$.

From (9) , $t = 1$ or 125. Take $y_0^3 = 1$, $z_0^3 = 125$, we have $y_0 = 1$, $z_0 = 5$.

Then the solutions are $1 + 5$, $\omega + 5\omega^2$, $\omega^2 + 5\omega$.

11. **(a)** $x^2 + px + q = 0 \Rightarrow \alpha + \beta = -p$, $\alpha\beta = q$ …. (1)

$(\alpha^3 - \beta)(\beta^3 - \alpha) = \alpha^3\beta^3 + \alpha\beta - (\alpha^4 + \beta^4) = (\alpha\beta)^3 + \alpha\beta - [(\alpha + \beta)^4 - 4\alpha\beta(\alpha^2 + \beta^2) - 6\alpha^2\beta^2]$

$= (\alpha\beta)^3 + \alpha\beta - \{(\alpha + \beta)^4 - 4\alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta] - 6(\alpha\beta)^2\} = q^3 + q - \{(-p)^4 - 4q[(-p)^2 - 2q] - 6q^2\}$, by (1)

$= q^3 + 2q^2 + q - (p^4 - 4p^2 q + 4q^2) = q(q + 1)^2 - (p^2 - 2q)^2$ …. (2)

One of the roots of the equation $x^2 + px + q = 0$ is equal to the cube of the other

$\Leftrightarrow \alpha^3 = \beta$ or $\beta^3 = \alpha \Leftrightarrow (\alpha^3 - \beta)(\beta^3 - \alpha) = 0 \Leftrightarrow q(q + 1)^2 - (p^2 - 2q)^2 = 0$, by (2)

$\Leftrightarrow (p^2 - 2q)^2 = q(q + 1)^2$.

(b) $x^2 + px + q = 0 \Rightarrow \alpha + \beta = -p$, $\alpha\beta = q$ …. (1)

$(t\alpha + \beta) + (t\beta + \alpha) = (1 + t)(\alpha + \beta) = (1 + t)(-p) = -p(1 + t)$

$(t\alpha + \beta)(t\beta + \alpha) = t(\alpha^2 + \beta^2) + \alpha\beta + t^2\alpha\beta = t[(\alpha + \beta)^2 - 2\alpha\beta] + \alpha\beta + t^2\alpha\beta = t[(-p)^2 - 2q] + q + t^2 q$

$= t^2 q + t(p^2 - 2q) + q$

Hence the new equation is $x^2 + p(1 + t)x + t^2 q + t(p^2 - 2q) + q = 0$

If $p^2 - 4q < 0$, $p \neq 0$, then (1) has two complex conjugate roots α and β .

Since p, q are real, and α and β are complex conjugates ,

(1) If $t \neq 1$ and is real, $(t\alpha + \beta)$, $(t\beta + \alpha)$ are complex numbers.

(2) If $t = 1$, $(t\alpha + \beta)$, $(t\beta + \alpha)$ are equal to $-p$.

Since $p \neq 0$, therefore $(t\alpha + \beta)$, $(t\beta + \alpha)$ are different from zero.

12. **(a)** p^2 and q are the roots of $x^2 + bx + q = 0$ …. (1)

$\therefore$ $p^4 + p^2 b + q = 0$ …. (2) , $q^2 + bq + q = 0$ …. (3)

Case (1), If $q \neq 0$, then from (3) , $q + b + 1 = 0 \Rightarrow q = -(b + 1)$ …. (4)

(4) $\downarrow$ (2), $p^4 + p^2 b + -(b + 1) = 0 \Rightarrow (p^2 + b + 1)(p^2 - 1) = 0$

$\therefore$ $p^2 = -(b + 1)$ or $p^2 = 1$

$\therefore$ $p = \pm\sqrt{-(b+1)}$ or $p = \pm 1$.

Case (2), If $q = 0$, then (2) becomes $p^4 + p^2 b = 0 \Rightarrow p^2 (p^2 + b) = 0$

$\therefore$ $p = 0$ or $\pm\sqrt{-b}$

(b) The equation $x^2 + bx + c = 0 \Rightarrow \alpha + \beta = -b$, $\alpha\beta = c$ …. (1)

$$\frac{1}{1+\alpha^2} + \frac{1}{1+\beta^2} = \frac{\alpha^2 + \beta^2 + 2}{(1+\alpha^2)(1+\beta^2)} = \frac{(\alpha+\beta)^2 - 2\alpha\beta + 2}{1 + (\alpha+\beta)^2 - 2\alpha\beta + (\alpha\beta)^2} = \frac{b^2 - 2c + 2}{1 + b^2 - 2c + c^2}$$

$$\frac{1}{1+\alpha^2}\frac{1}{1+\beta^2} = \frac{1}{1 + b^2 - 2c + c^2}$$

$\therefore$ The new equation is $(1 + b^2 - 2c + c^2)x^2 + (2c - 2 - b^2)x + 1 = 0$.

If $b = 1$ and $c = -1$, then the given equation is $x^2 + x - 1 = 0$, with roots $x = \dfrac{-1 \pm \sqrt{5}}{2}$

The new equation is $5x^2 - 5x + 1 = 0$, with roots $x = \dfrac{5 \pm \sqrt{5}}{10}$.

13. Let $\frac{\alpha}{r}, \alpha, \alpha r$ be the roots of $x^3 + 3x^2 + bx + c = 0$ …. (1)

Then $\alpha\left(\frac{1}{r} + 1 + r\right) = -3$ …. (2) , $\alpha^2\left(\frac{1}{r} + 1 + r\right) = b$ …. (3) , $\alpha^3 = -c$ …. (4)

(3)/(2), $\alpha = -\frac{b}{3}$ …. (5) , (5)$\downarrow$(4), $c = \frac{b^2}{27}$ …. (6)

(6)$\downarrow$(1), $x^3 + 3x^2 + bx + \frac{b^2}{27} = 0$. From (5) and division, $\left(x + \frac{b}{3}\right)\left[x^2 + \left(3 - \frac{b}{3}\right)x + \frac{b^2}{9}\right] = 0$

$$\therefore \quad x = -\frac{b}{3} \text{ or } x = \frac{-\left(3 - \frac{b}{3}\right) \pm \sqrt{\left(3 - \frac{b}{3}\right)^2 - \frac{4b^2}{9}}}{2} \quad \text{…. (7)}$$

In (7), $\Delta = \left(3 - \frac{b}{3}\right)^2 - \frac{4b^2}{9} = 9 - 2b + \frac{b^2}{9} - \frac{4b^2}{9} = \frac{27 - 6b - b^2}{3}$ …. (8)

From (5), Since α is an integer and $\alpha \neq 0$, b must be a non-zero integer divisible by 3 …. (9)

In (8), Since the roots are integers, $\Delta > 0$

$\Rightarrow b^2 + 6b - 27 = (b + 9)(b - 3) < 0 \Rightarrow -9 < b < 3$ …. (10)

The only b satisfying (9) and (10) is that $b = -6$ and $\Delta = 9$.

From (7), $x = 2, -1, -4$.

14. $x^4 + a_3 x^3 + a_2x^2 + a_1x + a_0 = 0$ …. (1)

z is a root of (1), $z^4 + a_3 z^3 + a_2z^2 + a_1z + a_0 = 0$ …. (2)

$1/z$ is a root of (1), $(1/z)^4 + a_3 (1/z)^3 + a_2(1/z)^2 + a_1(1/z) + a_0 = 0$

$\Rightarrow \quad a_0z^4 + a_1z^3 + a_2z^2 + a_3z + 1 = 0 \quad \Rightarrow \quad z^4 + \left(\frac{a_1}{a_0}\right)z^3 + \left(\frac{a_2}{a_0}\right)z^2 + \left(\frac{a_3}{a_0}\right)z + \left(\frac{1}{a_0}\right) = 0$ …. (3)

Since z and $1/z$ are roots, (2) and (3) are identical.

By comparing coefficients of constant terms of (2) and (3), $a_0 = \frac{1}{a_0} \Rightarrow a_0{}^2 = 1 \Rightarrow a_0 = \pm 1$

Case (1), When $a_0 = 1$, $a_1 = a_3$ (by comparing coeffs of x^2-terms)

Case (2), When $a_0 = -1$, $a_1 = -a_3$, $a_2 = 0$ (by comparing coeffs of x^3-terms and x^2-terms)

$\therefore$ The necessary and sufficient conditions are : $[(a_0 = 1, \quad a_1 = a_3)$ or $(a_0 = -1, \quad a_1 = -a_3, \quad a_2 = 0)]$

Put $p(x) = x^4 + a_3 x^3 + a_2x^2 + a_1x + a_0$

For Case (2), $p(1) = 1 + a_3 + a_2 + a_1 + a_0 = 0$, $p(-1) = 1 - a_3 + a_2 - a_1 + a_0 = 0$.

$\therefore$ $(x - 1)(x + 1)$ is a factor of $p(x)$ by Factor Theorem.

Division of $p(x)$ by $(x - 1)(x + 1)$ gives a quadratic factor and the problem reduces to solving quadratic equation.

For Case (1), equation (1) reduces to $x^4 + a_1 x^3 + a_2x^2 + a_1x + 1 = 0$

$\Rightarrow x^2 + a_1x + a_2 + \frac{a_1}{x} + \frac{1}{x^2} = 0 \Rightarrow \left(x^2 + \frac{1}{x^2}\right) + a_1\left(x + \frac{1}{x}\right) + a_2 = 0$ …. (4)

Put $y = x + \frac{1}{x}$ …. (5)

Then $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2 = y^2 - 2$

(4) becomes $y^2 - 2 + a_1y + a_2 = 0$ or $y^2 - 2 + a_1y + (a_2 - 2) = 0$ …. (6)

(6) is a quadratic and let y_1 and y_2 be the roots. From (5), we have:

$x^2 - y_1x + 1 = 0$ and $x^2 - y_2x + 1 = 0$ …. (7)

which are quadratic equations in x and (7) gives the roots of (1).

15. $x^4 - s_1x^3 + s_2x^2 - s_3x + s_4 = 0$ …. (1)

Writing Σ as symmetric sum, then

$\Sigma\alpha = s_1$ …. (2) , $\Sigma\alpha\beta = s_2$ …. (3) , $\Sigma\alpha\beta\gamma = s_3$ …. (4) , $\alpha\beta\gamma\delta = s_4$ …. (5)

$\Sigma(\alpha\beta + \gamma\delta) = \Sigma\alpha\beta = s_2$ …. (6)

$\Sigma(\alpha\beta + \gamma\delta)(\alpha\gamma + \beta\delta) = \Sigma\alpha^2\beta\gamma = (\Sigma\alpha)(\Sigma\alpha\beta\gamma) - 4\alpha\beta\gamma\delta = s_1 s_3 - 4 s_4$ …. (7)

$(\alpha\beta + \gamma\delta)(\alpha\gamma + \beta\delta)(\alpha\delta + \beta\gamma) = \Sigma\alpha^3\beta\gamma\delta + \Sigma\alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma\delta)\Sigma\alpha^2 + [(\Sigma\alpha\beta\gamma)^2 - 2\Sigma\alpha^2\beta^2\gamma\delta]$

$= (\alpha\beta\gamma\delta)[(\Sigma\alpha)^2 - 2\Sigma\alpha\beta] + [(\Sigma\alpha\beta\gamma)^2 - 2\alpha\beta\gamma\delta\,\Sigma\alpha\beta] = s_4[s_1^2 - 2s_2] + [s_3^2 - 2s_4 s_2] = -(4s_2 s_4 - s_1^2 s_4 - s_3^2)$….(8)

From (6), (7), (8), the required cubic equation is

$$y^3 - s_2y^2 + (s_1 s_3 - 4 s_4)y + (4s_2 s_4 - s_1^2 s_4 - s_3^2) = 0 \quad \ldots. \quad (9)$$

Supposing that methods of solving cubic equation are known,

(9) is solved and let the roots be y_1, y_2 and y_3 . Then:

$\alpha\beta + \gamma\delta = y_1$ …. (10) , $\alpha\gamma + \beta\delta = y_2$ …. (11) , $\alpha\delta + \beta\gamma = y_3$ …. (12)

From (5), $(\alpha\beta)(\gamma\delta) = s_4$ …. (13)

From (10) and (13), $\alpha\beta$ and $\gamma\delta$ are roots of the quadratic equation : $z^2 - y_1z + s_4 = 0$ …. (14)

Suppose (14) is solved, let the roots be $z_1 = \alpha\beta$, $z_2 = \gamma\delta$ …. (15)

From (3), $\gamma\delta(\alpha + \beta) + \alpha\beta(\gamma + \delta) = s_3 \Leftrightarrow z_2 (\alpha + \beta) + z_1 (\gamma + \delta) = s_3$ …. (16)

From (2), $(\alpha + \beta) + (\gamma + \delta) = s_1$ …. (17)

(17) and (18) are simultaneous equation with unknowns $(\alpha + \beta)$, $(\gamma + \delta)$.

Let the equations be solved and $k_1 = \alpha + \beta$, $k_2 = \gamma + \delta$ …. (18)

(15) , (16) form two quadratic equations for which $\alpha, \beta, \gamma, \delta$ can be found.

Note that if (11) or (12) is used instead of (10) , the solution is the same due to the arbitrary use of the notation $\alpha, \beta, \gamma, \delta$ for the four roots of (1).

16. **(a)** We use induction on k that $x^k + x^{-k}$ can be expressed as a polynomial $p_k(z)$ in $z = x + x^{-1}$.

The proposition is obviously true for $k = 1$ where $p_1(z) = z$.

For $k = 2$, $x^2 + x^{-2} = (x + x^{-1})^2 - 1 = z^2 - 2$ and the proposition is true for $k = 2$.

Assume the proposition is true for $k \le n$ where $n \in \mathbb{N}$.

For $k = n + 1$, $x^{(n+1)} + x^{-(n+1)} = (x^n + x^{-n})(x + x^{-1}) - [x^{(n-1)} + x^{-(n-1)}]$

$= p_n(z)\, p_1(z) - p_{n-1}(z)$, by inductive hypothesis.

Putting $p_{n+1}(z) = p_n(z)\, p_1(z) - p_{n-1}(z)$, then the proposition is also true for $k = n + 1$.

By the Principle of Mathematical Induction, the proposition is true $\forall k \in \mathbb{N}$.

(b) $x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1 = 0$ …. (1)

$(x^3 + x^{-3}) + a(x^2 + x^{-2}) + b(x + x^{-1}) + c = 0$ …. (2)

Putting $z = x + x^{-1}$, then (1) becomes $(z^3 - 3z) + a + bz + c = 0$.

or $z^3 = az^2 + (b - 3)z + (c - 2a) = 0$ …. (3)

$\therefore$ If α is a root of the polynomial equation (1), then $\alpha + \alpha^{-1}$ is a root of (2) .

(c) $x^4 + 4x^3 + 5x^2 + 4x + 1 = 0 \Leftrightarrow (x^2 + x^{-2}) + 4(x + x^{-1}) + 5 = 0$ …. (4)

Putting $z = x + x^{-1}$, then (4) becomes $(z^2 - 2) + 4z + 5 = 0$

or $z^2 + 4z + 3 = 0 \Leftrightarrow (z + 3)(z + 1) = 0$

$z = x + x^{-1} = -3$ or $z = x + x^{-1} = -1$

$x^2 + 3x + 1 = 0$ or $x^2 + x + 1 = 0$

$x = \dfrac{-3 \pm \sqrt{5}}{2}$ or $x = \dfrac{-1 \pm \sqrt{3}i}{2}$

17. $x^3 + 3qx + r = 0 \quad (r \neq 0)$ …. (1)

$\alpha + \beta + \gamma = 0$ …. (2) $\qquad \alpha\beta + \beta\gamma + \gamma\alpha = 3q$ …. (3) $\qquad \alpha\beta\gamma = -r$ …. (4)

Put $\quad p(x) = r^2(x^2 + x + 1)^3 + 27q^3x^2(x+1)^2$

Then $\quad p\left(\frac{\alpha}{\beta}\right) = r^2\left(\frac{\alpha^2}{\beta^2} + \frac{\alpha}{\beta} + 1\right)^3 + 27q^3\frac{\alpha^2}{\beta^2}\left(\frac{\alpha}{\beta}+1\right)^2 = r^2\left(\frac{\alpha^2+\alpha+1}{\beta^2}\right)^3 + 27q^3\frac{\alpha^2}{\beta^2}\frac{(\alpha+\beta)^2}{\beta^2}$

$$= r^2\left(\frac{\alpha^3-\beta^3}{\beta^2(\alpha-\beta)}\right)^3 + 27q^3\frac{\alpha^2}{\beta^4}(-\gamma)^2 = r^2\left(\frac{(-3q\alpha+r)+(-3q\beta+r)}{\beta^2(\alpha-\beta)}\right)^3 + 27q^3\frac{\alpha^2\gamma^2}{\beta^4}$$

$$= r^2\left(\frac{-3q}{\beta^2}\right)^3 + 27q^3\frac{(\alpha\beta\gamma)^2}{\beta^6} = -\frac{27q^3r^2}{\beta^6} + 27q^3\frac{(-r)^2}{\beta^6} = 0$$

$\therefore \quad \frac{\alpha}{\beta}$ satisfies $\quad p(x) = r^2(x^2+x+1)^3 + 27q^3x^2(x+1)^2 = 0$ …. (5)

(a) When $q = 0$, (1) becomes $x^3 + r = 0$ …. (6)

The roots of (6) are $\sqrt[3]{-r}$, $\sqrt[3]{-r}\omega$, $\sqrt[3]{-r}\omega^2$, where ω is the complex cube roots of unity.

Take α be any one of these three roots and β be any other root not equal to α .

Then $\quad \alpha = \omega\beta$ or $\alpha = \omega^2\beta$ …. (7)

When $q = 0$, (5) becomes $x^2 + x + 1 = 0 \quad (r \neq 0)$ and the roots are ω, ω^2 .

$\therefore \quad \frac{\alpha}{\beta} = \omega$ or $\frac{\alpha}{\beta} = \omega^2$ …. (8)

It can be seen that (7) and (8) are the same.

(b) If $4q^3 + r^2 = 0$, then $r^2 = -4q^3$ …. (9)

(9) $\downarrow$ (5), $\quad p(x) = -4q^3(x^2+x+1)^3 + 27q^3x^2(x+1)^2 = 0$ …. (10)

If $q = 0$, then by (9) , $r = 0$, contradicting to $r \neq 0$.

$\therefore \quad q \neq 0$ and (10) becomes $-4(x^2+x+1)^3 + 27x^2(x+1)^2 = 0$

$\Leftrightarrow \quad 4x^6 + 12x^5 - 3x^4 - 26x^3 - 3x^2 + 12x + 4 = 0 \quad \Leftrightarrow \quad (x-1)^2(x+2)^2(2x+1)^2 = 0$

Since $\frac{\alpha}{\beta}$ satisfies (10), $\quad \frac{\alpha}{\beta} = 1$ or -2 or $-\frac{1}{2}$.

(i) If $\frac{\alpha}{\beta} = 1$, then $\alpha = \beta$.

(ii) If $\frac{\alpha}{\beta} = -2$, then $\alpha = -2\beta$. From (1), $-2\beta + \beta + \gamma = 0$, $\beta = \gamma$.

(iii) If $\frac{\alpha}{\beta} = -\frac{1}{2}$, then $\beta = -2\alpha$. From (1), $\alpha - 2\alpha + \gamma = 0$, $\alpha = \gamma$.

In any one of the above cases, there is a double root for (1).

18. **(a)** A.M. – G.M. : Let x_i (i = 1, 2, …., n) be n distinct positive numbers, then $\frac{\sum x_i}{n} > \Pi x_i$.

(b) $f(x) \equiv x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_{n-1}x + a_n = 0$ has n distinct positive roots α_i (i = 1, 2, …., n)

Then $\sum \alpha_1 = -a_1$ …. (1) , $\sum \alpha_1\alpha_2 = a_2$ …. (2) , ….,

$$\sum \alpha_1\alpha_2\ldots\alpha_{n-1} = (-1)^{n-1} a_{n-1} \quad \ldots. \quad (n-1) \quad , \quad \alpha_1\alpha_2\ldots\alpha_{n-1}\alpha_n = (-1)^n a_n \ \ldots. \ (n)$$

Writing $a_i = (-1)^i \binom{n}{i} b_i{}^i$, then $a_n = (-1)^n \binom{n}{n} b_n{}^n = (-1)^n b_n{}^n$ …. (i)

$$a_{n-1} = (-1)^{n-1}\binom{n}{n-1} b_{n-1}{}^{n-1} = (-1)^{n-1} n b_{n-1}{}^{n-1} \quad \ldots. \quad (ii)$$

From (i) and (n), $b_n{}^n = \alpha_1\alpha_2\ldots\alpha_{n-1}\alpha_n$ …. (iii)

From (ii) and (n-1), $n b_{n-1}{}^{n-1} = \sum \alpha_1\alpha_2\ldots\alpha_{n-1}$ …. (iv)

From (iii), $b_n = (\alpha_1\alpha_2\ldots\alpha_{n-1}\alpha_n)^{1/n} = \left[(\Pi \alpha_1\alpha_2\ldots\alpha_{n-1})^{\frac{1}{n-1}} \right]^{\frac{1}{n}} = \left[(\Pi \alpha_1\alpha_2\ldots\alpha_{n-1})^{\frac{1}{n}} \right]^{\frac{1}{n-1}}$

$$< \left[\frac{\sum \alpha_1\alpha_2\ldots\alpha_{n-1}}{n} \right]^{\frac{1}{n-1}} < \left[b_{n-1}{}^{n-1} \right]^{\frac{1}{n-1}} = b_{n-1} \quad \ldots. \quad (v)$$

$$f'(x) \equiv nx^{n-1} + (n-1)x^{n-2} + \ldots + 2a_{n-2}x + a_{n-1} = n\left[x^{n-1} + \left(\frac{n-1}{n}\right)x^{n-2} + \ldots + \frac{2}{n}a_{n-2}x + \frac{1}{n}a_{n-1} \right]$$

Consider f '(x) = 0.

Since f(x) = 0 has n distinct positive real roots and between any two roots of f(x) = 0 , there is at least one real root of f'(x) = 0, where deg f'(x) = n – 1 .

∴ f '(x) = 0 has (n – 1) distinct positive real roots, each lying between two roots of f(x) = 0 .

Let $\beta_1, \beta_2, \ldots, \beta_{n-1}$ be the (n – 1) distinct positive real roots of f '(x) = 0

$$\sum \beta_1\beta_2\ldots\beta_{n-2} = (-1)^{n-2}\frac{2}{n}a_{n-2} = (-1)^{n-2}\frac{2}{n}\left[(-1)^{n-2}\binom{n}{n-2} b_{n-2}{}^{n-2} \right] = \frac{2}{n}\frac{n(n-1)}{2} b_{n-2}{}^{n-2} = (n-1) b_{n-2}{}^{n-2}$$

…. (vi)

$$\beta_1\beta_2\ldots\beta_{n-1} = (-1)^{n-1}\frac{a_{n-1}}{n} = (-1)^{n-1}\frac{1}{n}\left[(-1)^{n-1} n b_{n-1}{}^{n-1} \right] = b_{n-1}{}^{n-1} \quad \ldots. \quad (vii)$$

$$b_{n-1} = (\beta_1\beta_2\ldots\beta_{n-1})^{\frac{1}{n-1}} = \left[(\Pi \beta_1\beta_2\ldots\beta_{n-2})^{\frac{1}{n-2}} \right]^{\frac{1}{n-1}} = \left[(\Pi \beta_1\beta_2\ldots\beta_{n-2})^{\frac{1}{n-1}} \right]^{\frac{1}{n-2}}$$

$$< \left[\frac{\sum \beta_1\beta_2\ldots\beta_{n-2}}{n-1} \right]^{\frac{1}{n-2}} = \left[\frac{(n-1) b_{n-2}{}^{n-2}}{n-1} \right]^{\frac{1}{n-2}} = b_{n-2} \quad \ldots. \quad (viii)$$

Similarly, by considering f''(x), f'''(x), …, $f^{(n-2)}(x)$, we get correspondingly,
$b_{n-2} > b_{n-3}$, $b_{n-3} > b_{n-4}$, … , $b_2 > b_1$,
Combining all these inequalities, we have $b_1 > b_2 > \ldots > b_{n-1} > b_n$.